EXPLORING STUDENTS' IMAGES AND DEFINITIONS OF AREA

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ABSTRACT. *This study examines several aspects of the images and definitions that eight students of high school (16 years old) have regarding area. The analysis is performed through 10 open questions to which students answered through written statements, drawings, concept maps. The protocols shed light on the ways used by students to communicate their ideas and on the role they ascribe to definitions in their mathematical experience.*

INTRODUCTION

In studying the various aspects of proof we became aware of the crucial role of definitions. In particular, we have often observed that some cause of failures are due to the role students ascribe to them and how they deal with them. What happens can be explained considering the statement to prove as having a hypertext structure: it contains 'hot words' that the student has to single out and on which to 'click' when necessary. The operation of clicking establishes a link between hot words, concept images and concept definitions (henceforth called images and definitions) behind them to get the information useful to go on in proving. If one of these procedures (recognising hot words, clicking on them, getting information) is not activated the statement is obscure and to prove is a hard task. The research reported in (Furinghetti & Paola, 1997) may be an exemplification of this hypertext metaphor. We have found that the statement 'Prove that the product of any three consecutive natural numbers is divisible by 6' resulted difficult to prove for most students (aged from 14-17) because they did not recognised that 'divisible' was a 'hot word' in the statement to prove or, when they did, were not able to use the definition of 'divisible' that has been introduced to them previously. In that case we had the impression that students did not ascribe cognitive value to definitions, they seemed to perceive them only as labels which are not relevant to the mathematical work. Other authors have observed analogous students' behaviour. Rasslan and Vinner (1998, p.33) write that the student 'does not necessarily use the definition when deciding whether a given mathematical object is an example or non example of the concept. In most cases, he or she decides on the basis of the concept image, that is, the set of all the mental pictures associated in his/her mind with the name of the concept, together with all the properties characterising them'. Bills and Tall (1998, p.105) have introduced the expression 'formally operable' for a given individual to indicate a (mathematical) definition or theorem which an individual is able to use 'in creating or (meaningfully) reproducing a formal argument'. We can say that for our students the definition of 'divisible' was not operable.

We quote the following passage, taken from Wheeler (1991, p.1), to summarise some important points on the role of definitions in proving:

'When we talk about proof, it seems very often that we shift quite unconsciously between talking of proof as something technically proven and proof as a sign for formalism. I wish we would give more attention to the business of definition rather than that of proof *per se*, or at least that we would really take into account of the act of making definitions, because if one's talking about the normalization of mathematics it seems to me that it's in the definitions that we find the vectors of mathematics: these are things that we choose to define this way because they have a future, because they will go somewhere, because we can do something with them. Now, proof when it's finished, is finished. Of course, there are proof techniques that we can apply in other cases, so I mustn't be unfair to proof and say that we can't do anything with it other than just prove one particular theorem, but definitions *always* have to be shaped in order to point us onwards, so that we can go somewhere, otherwise they would be bad definitions that we would abandon, that somebody subsequently would change'.

OUR STUDY: METHODOLOGY AND AIMS

On the ground of the previous observations we consider very important to explore students' behaviours in defining and to promote classroom activities which lead students to reflect on definitions. In the present paper we report an activity of this kind centered on the concept of area. In carrying out this activity our aim was twofold: from one hand it was to investigate which images students have elaborated of the concept of area and how make them explicit, from the other hand our aim was to promote cognitive activities about area. The concept considered is recognised as very difficult, see (Douady & Perrin, 1989), but is not studied so much from the educational point of view, in particular at the level of high school.

The population is constituted by eight 16 years old students of a Scientific Lyceum, an Italian high school in which mathematics plays an important role. They were requested to answer the questions reported in Table 1 (next page), using written statements, drawings, concept maps. We explained them our purposes and asked for an active collaboration. They worked with good willingness and fulfilled our expectation. The allowed time was 50 minutes.

We tried to structure the questionnaire in such a way that students' thoughts and possible inconsistencies could emerge. Question Q.1 is aimed at verifying whether students' images of plane regions are only based on elementary patterns (polygons, circle) or include any kind of shapes. The distinction is significant since in the first case students may have difficulties in thinking to situations in which a formula to compute area based on elementary operations does not exist. Question Q.2 is aimed at outlining the nature of the concept in question through links established with other mathematical concepts. Questions Q.3, Q.4, Q.5 and Q.6 were conceived to stress the specificity of the languages used in different school situations. These questions were inspired by Austin and Howson (1979) who observe that in classroom there are different mathematical languages: the language used with mates, with the teacher and the language of mathematics. The remaining questions are mainly aimed at orienting students towards activities of metacognition, that is to say, using the Schoenfeld (1987) expression 'thinking about thinking'. In particular Question Q.8 is focused on the possible origin of students' misunderstandings. Question Q.9 is addressed to see whether new forms of representation of knowledge may act as a stimulus to make explicit concept images.

Q. 1. When one talks about the *area* of a plane figure, which images are evoked to you? Represent some of them (three or four) in your protocol.
Q. 2. Which parts and which topics of mathematics do you link to the notion of *area of a plane figure*? Write in your protocol and try to give an explicatory example for each issues you refer to.
Q. 3. To explain to a fourth-fifth grade student what *area of a plane figure* means, what would you say?
Q. 4. To explain to an eighth grade student what area of a plane figure means, what would you say?
Q. 5. To explain to a mate of yours what *area of a plane figure* means, what would you say?
Q. 6. To say to your teacher what *area of a plane figure* means for you, what would you say?
Q. 7. Are you surprised about the questions 3, 4, 5, 6? In other words is it possible to use different characterisations of the same concepts according on the person you are addressing to?
Q. 8. Are there analogies and differences between the use and the meaning of the word *area* in the common and in the mathematical language.
Q. 9. Construct a concept map about the concept of *area*.
Q. 10. Reconstruct the didactic path followed in your school career until now as for the concept of *area*.

Table 1. The questionnaire on the students' images and definitions of area

ANALYSIS OF THE ANSWERS

In analysing the protocols we have singled out key elements and orientations. We have reported them in Table 2 (which takes the next two pages) to give a synoptic view of answers. In abbreviating and translating sentences some nuances of meanings are lost, but we feel that the basic issues of the students' reactions are kept. Table 2 has two entries: reading horizontally we have the answers to the same question by the eight students, reading vertically we have the picture of each student's images and definitions as emerged from protocols. In the following we give our interpretation of the findings reported in Table 2.

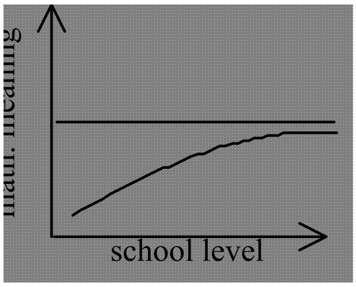
Question Q.2 offers insights on the *students' difficulties* about area. Only two students (Protocols 3 and 6) mention the 'measure'. Nobody refers to real numbers, which for us was the most obvious association. Another missing link for all students except one (Protocol 7) is that with the concept of function. In comparing the answers to Q.1 and Q.2 we find inconsistencies (see rows referring to Q.1 and Q.2 in Table 2) which are generated by the passage from the simple geometrical/visual context to richer contexts encompassing arithmetic, algebra, analytical geometry. In Protocols 1, 2, 5 the geometrical shapes mentioned in the row of Q.1 are not in relation with the answers to Q.2. In principle, we consider important to encourage students to go across different parts of mathematics: this could promote the flexibility advocated by Tall and Gray (1993). Our findings show that this process needs of careful control. Going across contexts does not always create meaning, in some cases may cloud the existing one so that what is generated is not fruitful communication between contexts, but rather *contamination of contexts*.

Questions Q.3, Q.4, Q.5, Q.6 shed light on *the evolution of the way used to communicate mathematical ideas according to the interlocutor*. At the primary level there are manifest references to concrete elements: 'space occupied by the figure' (Protocol 2), 'quantity of substance covering up the figure' (Protocol 3), 'to hatch the figure' (Protocol 4), 'to colour the figure with pencil' (Protocol 8). There is a precise use of efficient 'ostensives' (hatching, colour). Things change when the school level increases: already in the statements addressed to eight grade students the language is more formal, which does not mean necessarily more precise, nor more oriented to generalisation and abstraction. Students do not add useful information, they only paraphrase the statements addressed to younger students eliminating ostensives and concrete ideas and adding negligible details. For example, in Protocol 2 the expression addressed to the fourth-fifth grade student 'space occupied by a figure', which evokes a physical situation, is substituted by 'portion of plane contained at the interior' when addressing to the eight grade student. The loss of spontaneity is evidenced in Protocol 8 where we observe a real escalation: - the ostensive for the fourth-fifth grade (coloured pencil), - the semi-intuitive for the eight grade ('what is contained in the segments') - the formal expression ('It is called'). It is curious to observe that to use the words 'It is called' was advocated by Smith (1911) in his famous treatise on mathematics teaching to 'mark the statement at once as a definition' (p.158), that is to say to distinguish it from a theorem. Protocol 8 evidences a different behaviour according to the age of the interlocutor: with the youngest the student tries to make understandable his message using the 'common sense', with the oldest and with the teacher he is only interested in conveying the idea that he is defining something, no matter if what he is saying is understandable or not. Another aspect of the evolution in the way of communicating according to the school level is the relationship general/generic/particular. In Protocol 8 the student adds the adjectives 'concave or convex' to the word 'figure' for mates and for the teacher. In Protocol 1 we find a similar behaviour: to the fourth-fifth grade student it is said 'portion of plane contained in *lines*', while for the older students the sentence becomes '[...] *segments or curved lines*' [emphasis is our].

As for Question Q.7 all students agree that there has to be a distinction between the languages used in the four different situations of Questions Q.3 to Q.6, but from their explanations we infer that the difference is in the form, not in the substance. Advancing in the age there is no gain in generality, abstraction or elegance, while there is loss in meaning. The creativity seems inhibited along the years; the students' behaviour can be referred to the ritual and symbolic schemes discussed by Harel and Sowder (1996) in their studies on proof.

Question Q.8 adds further information *about the relationship between mathematical knowledge and common sense*. In Protocol 1 it is pointed out that in the common language the area is considered *only* [emphasis is our] as a measure. The student is not able to accept the right suggestions coming from the common sense, even more it seems that the common sense is perceived as something against mathematics.

Analysing our findings we feel that the students' capacity to grasp mathematical meaning through personal and autonomous elaboration of ideas has the asymptotic trend illustrated in the figure beside. At the age of the students examined (16 years) the upper bound seem to be already reached. In Protocol 2 we found the statement



'At the lyceum the teacher tells us where formulas come from', which is not the same as saying that the teacher explains how to find formulas. Freudenthal (1973) has criticised the teachers' practice of providing students with definitions given a priori instead of constructing them together with students. May we interpret the student's statement as a criticism in this direction?

Concept maps

At the moment of answering questions students did not know what 'concept map' means. We simply explained them that it is 'a graphical representation of domain material generated by the learner in which nodes are used to represent domain key concepts, and links between them denote the relationship between these concepts', see Jones (1998, p.161). To analyse the students' protocols we have considered:
- the number of issues reported in the concept map
- if arrows are single or double, how arrows go in and out at the nodes
- the presence and the kind of explanations of links
- the presence of the word definition or its derivative
- the type of iconic representations used (only words, words in boxes, etc.)

To single out the differences when using the verbal language or concept maps we have compared the number and the type of issues in the answers to Q.2 (that about the links with parts of mathematics) and in the concept map. Protocol 4 has the same number of issues (six, the highest among the eight students examined) and of the same type in both cases. The concept map of this protocol is very rich and it is the only in which links have a written explanation. In Protocol 7 there is the same number of issues (four) in Q.2 and in the concept map, but the issues mentioned are different. In Protocol 5 there is one issue in the answer to Q.2 and there is not the concept map. In the other protocols the number of issues appearing is higher in concept map than in Q.2.

About the different styles we can distinguish:
- those who simple translate into the iconic form their word statements (for example, Protocol 1)
- those having a linear pattern to approach the concept ((for example, Protocol 3). The arrows are single, at most one arrow arrives to and leaves from nodes. The focus is on the order of facts more than on links between them. The word definition appears among the nodes
- those who really add information making the links more explicit and finding new issues ((for example, Protocol 4).

PRELIMINARY CONCLUSIONS

The concept of area, through its epistemological and conceptual aspects and its character of being a cross-roads of different parts of mathematics has revealed itself suitable to show that students have in their mind a jungle of concept images, concept definitions, which are not completely under their control. Moreover for them the problem is not only to elaborate images which can flow into definitions mathematically acceptable, but to find means which may make them explicit, consistent and clear to others and to themselves. From our study we have obtained a number of indications for the classroom practice as well as topics which deserve attention for future research. A point we would like to stress as a first preliminary conclusion is the importance of making students to reflect not only on their way of thinking, but also on the way of representing their thoughts, so that to the Schoenfeld's expression 'thinking about thinking' we may add its paraphrase 'thinking about representing'.

REFERENCES

Austin, J. L. & Howson, A. G.: 1979, 'Language and mathematics education', *Educational studies in mathematics*, v.10, 161-197.

Bills, L. & Tall, D.: 1998, 'Operable definitions in advanced mathematics: the case of the least upper bound', in A. Olivier & K. Newstead (editors), *Proceedings of PME 22*, v.2, 104-11.

Douady, R. & Perrin, M.-J.: 1989, 'Un processus d'apprentissage du concept d'aire plane', *Educational studies in mathematics*, v.29, 387-424.

Freudenthal, H.: 1973, *Mathematics as an educatonal task*, Dordrecht, Reidel.

Furinghetti, F. & Paola, D.: 1997, 'Shadows on proof', in E. Pekonen (editor), *Proceedings of PME 21*, v.2, 273-280.

Gray, E. M. & Tall, D.: 1993, 'Success and failure in mathematics: the flexible meaning of symbols as process and concept', *Mathematics teaching*, n.142, 6-10.

Harel, G. & Sowder, L.: 1996, ‘Classifying processes of proving’, in A. Gutierrez & L. Puig (editors), *Proceedings of PME 20* (Valencia), v.3, 59-65.
Jones, K.: 1998, ‘Mathematics graduates conceptions of mathematical proof’, in *Pre-Proceedings of ICMI Study Conference on the teaching and learning of mathematics at university level*, NIE, Singapore, 161-164.
Schoenfeld, A. (editor): 1987, *Cognitive science and mathematics education*, Erlbaum, Hillsdale.
Smith, D. E.: 1911, *The teaching of geometry*, Ginn and company, Boston ecc.
Vinner, S.: 1998, ‘Images and definitions for the concept of increasing/decreasing function’, in A. Olivier & K. Newstead (editors), *Proceedings of PME 22*, v.4, 33-40.
Wheeler, D.: 1990, ‘Aspects of mathematical proof’, *Interchange*, v.21, n.1, 1-5.